A NOTE ON KASPAROV PRODUCT AND DUALITY

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Abstract. Using Paschke-Higson Duality [Hig][Pa], we get a natural index pairing $K_i(A)\times K_{i+1}(D_\Phi) \to \boldsymbol{Z}$ where $i = 0, 1(\text{mod}2)$ and A is a separable C^*-algebra, Φ is a representation of A on a separable Hilbert space $\mathbb{H}$. We prove this is a special case of Kasparov Product [Kas].

1. Introduction

The purpose of this paper is to give a clear exposition of a connection between KK-theory [Kas] and index pairing

$$K_i(A) \times K_{i+1}(D_\Phi) \to \mathbb{Z} \quad \text{where} \quad i = 0, 1(\text{mod}2)$$

which is defined in Section2. In fact, we are going to show that each index pairing is a Kasparov product only using elementary ingredients of K-theory and KK-theory. (see proposition 3.1 below and lemma 2.9, 2.12 below.) As an application of this approach, we show an alternate proof of Bott periodicity in KK-theory [Kas]; cf. Theorem 18.10.2 in [Bl]. In this proof, we do not use geometric argument (for example, use of Clifford algebra [Kas]) but operator theory and pure algebra.

2. Paschke-Higson Duality and Index pairing

In this section, we review Paschke-Higson duality theory [Hig]. Throughout this article, $\mathbb{H}$ is a separable infinite dimensional Hilbert space. $\mathcal{B}(\mathbb{H})$ is the set of linear bounded operators on $\mathbb{H}$. $\mathcal{K}(\mathbb{H})$(shortly $\mathcal{K}$) is an ideal of compact operators on $\mathbb{H}$, and $\mathcal{Q}(\mathbb{H})$(shortly $\mathcal{Q}$) is the Calkin algebra.

We use the following notation : if X and Y are operators in $\mathcal{B}(\mathbb{H})$ we shall write

$$X \sim Y$$

if X and Y differ by a compact operator.

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Note that every *-representation $\Phi : A \to \mathcal{B}(\mathbb{H})$ determines a *-homomorphism $\dot{\Phi}$ of A into the Calkin algebra.

Definition 2.1. Let A be C^*-algebra. A *-representation $\Phi : A \to B(\mathbb{H})$ is called *admissible* if it is non-degenerate and $ker(\dot{\Phi}) = 0$.

Remark 2.2. If a *-representation is *admissible*, then it is necessarily faithful and its image contains no non-zero compact operator.

Definition 2.3. Let Φ be a *-representation of A on $\mathbb{H}$. Denote by $D_\Phi(A)$ the *essential commutant* of $\Phi(A)$ in $\mathcal{B}(\mathbb{H})$. Thus

$$D_\Phi(A) = \{T \in \mathcal{B}(\mathbb{H}) \mid \forall a \in A, [\Phi(a), T] \sim 0\}$$

Given two representations Φ_0, Φ_1 on $\mathbb{H}_0, \mathbb{H}_1$ respectively, we say they are *approximately unitarily equivalent* if there exists a sequence $\{U_n\}$ consisting of unitaries in $\mathcal{B}(\mathbb{H}_0, \mathbb{H}_1)$ such that

$$Ad(U_n)\Phi_0(a) \sim \Phi_1(a) \quad \text{for all} \quad a \in A$$
$$\|Ad(U_n)\Phi_0(a) - \Phi_1(a)\| \to 0 \quad \text{as} \quad n \to \infty$$

We write $\Phi_0 \sim_u \Phi_1$ in this case.

Theorem 2.4. *(Voiculescu) Let A be a separable C^*-algebra and $\Phi_i : A \mapsto \mathcal{B}(\mathbb{H}_i)$ $i = 0, 1$ be non-degenerate *-representations. Then if Ker $\dot{\Phi}_0 \subset$ Ker Φ_1, then $\Phi_0 \oplus \Phi_1 \sim_u \Phi_0$ where $\dot{\Phi}$ is the natural *-homomorphism into the Calkin algebra induced by a *-representation Φ.*

Proof. See Corollary 1 in p343 of [Ar]. □

Corollary 2.5. *Assume $\Phi_i : A \mapsto \mathcal{B}(\mathbb{H}_i)$ are* admissible *representations for $i = 0, 1$. Then $\Phi_0 \sim_u \Phi_1$.*

Proof. Admissibility implies Ker$\dot{\Phi}_i$= 0. From this, the result is straightforward. □

Recall that an extension of a unital separable C^*-algebra A is a unital *-monomorphism τ of A into the Calkin algebra. We say τ is *split* if there is a non-degenerate *-representation ρ such that $\dot{\rho} = \tau$ and *semi-split* if there is a completely positive map ρ such that $\dot{\rho} = \tau$.

Corollary 2.6. *Let A be a separable unital C^*-algebra. If τ is a unital injective extension of A and if σ is a split unital extension of A, then $\tau \oplus \sigma$ is unitarily equivalent τ.*

Proof. See p352-353 in [Ar]. □

Now we will prove the existence of *admissible* representation of A.

Proposition 2.7. *There is a non-degenerate *-representation Φ of for a separable C^*-algebra A such that $Ker(\dot{\Phi}) = 0$.*

Proof. Let π be a faithful representation of A on $\mathbb{H}_\pi$. Take Φ as $\pi^\infty = \pi \oplus \pi \oplus \cdots$ and $\mathbb{H} = \mathbb{H}_\pi \oplus \mathbb{H}_\pi \oplus \cdots$ $\square$

Definition 2.8. When Φ, Ψ are *admissible* representations of A on $\mathbb{H}$, $D_\Phi(A)$ is isomorphic to $D_\Psi(A)$ by Corollary 2.5. Therefore we define $D(A) = D_\Phi(A)$ as the dual algebra of A up to unitary equivalence.

If p is a projection in $D_\Phi(A)$, we call it *ample* and can define an extension

$$\tau = \tau_{\Phi,p} : A \xrightarrow{p\Phi(\bullet)p = \Phi_p} \mathcal{B}(p\mathbb{H}) \xrightarrow{\pi} \mathcal{Q}(p\mathbb{H})$$

In general, this extension is not injective. However, if the extension $\tau_{\Phi,p}$ is injective, we call that an abstract Toeplitz extension.

To define the map from $K_0(D(A))(= K_0(D_\Phi(A)))$ to $Ext^{-1}(A)$, we need the following two technical lemmas.

Lemma 2.9. *Let A be a unital C^*-algebra. For any $\alpha \in K_0(D(A))$, there exists a ample projection $p \in D(A)$ such that $\alpha = [p]_0$.*

Proof. Step1: By Corollary 2.6, there is a unitary $u \in \mathcal{B}(p\mathbb{H} \oplus \mathbb{H}, \mathbb{H})$ such that Ad $(u)(\Phi_p \oplus \Phi)(a) \sim \Phi(a)$ for any $a \in A$ if p is *ample.* Let $U = \binom{p}{0}u$. We can easily check that $U \in \mathbb{M}_2(D_\Phi(A))$ and $UU^* = \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix}, U^*U = \begin{pmatrix} p & 0 \\ 0 & I \end{pmatrix}$. Therefore we have $[p \oplus I]_0 = [p \oplus 0]_0$. This implies $[p]_0 + [I]_0 = [P]_0$. In particular, $[I]_0 = 0$. Also you can conclude every two *ample* projections are Murray-von Neumann equivalent.
Step2: Note that $p \oplus I$ is always *ample* whether p is ample or not because $(\Phi \oplus \Phi)_{p \oplus I}(a)$ is never compact unless $a \in A$ is zero.
Step3: Any element in $K_0(D_\Phi(A))$ is written by $[q]_0 - [I_n]_0$ for some $q \in \mathbb{M}_n(D_\Phi(A))$. As we observed in Step1, this is just $[q]_0$. By Step2, we may assume q is ample for $\underbrace{\Phi \oplus \Phi \oplus \cdots \oplus \Phi}_{n}$. Now if we can show $[q]_0 = [p]_0$ for some *ample* $p \in D_\Phi(A)$, we are done.
Since $\underbrace{\Phi \oplus \Phi \oplus \cdots \oplus \Phi}_{n} \sim_u \Phi$, there exists $v : \mathbb{H}^n \mapsto \mathbb{H}$ s.t.

(1) $v^*v = 1_{B(\mathbb{H}^n)}, vv^* = 1_{B(\mathbb{H})}$

(2) $\mathrm{Ad}(v) \underbrace{\Phi \oplus \Phi \oplus \cdots \oplus \Phi}_{n}(a) - \Phi(a) \in \mathcal{K} \quad \forall a \in A.$

Let $V = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} v \in \mathbb{M}_n(D_\Phi(A))$. Then $[q]_0 = [VqV^*]_0 = [vqv^*]_0$. It's left to the reader to check that vqv^* is also *ample*. □

Lemma 2.10. *Let $\phi : A \to \mathcal{B}(\mathbb{H}_1 \oplus \mathbb{H}_2)$ be a *-representation. Write $\phi(a) = \begin{pmatrix} \phi_{11}(a) & \phi_{12}(a) \\ \phi_{21}(a) & \phi_{22}(a) \end{pmatrix}$. Suppose ϕ_{11} is *-homomorphism modulo $\mathcal{K}(\mathbb{H}_1)$. i.e., $\dot{\phi}_{11}$ is *-homomorphism. Then $\phi_{12}(a)$, $\phi_{21}(a)$ are compacts for any $a \in A$ and $\dot{\phi}_{22}$ is *-homomorphism.*

Proof. Using $\phi(aa^*) = \phi(a)\phi(a^*)$ with decomposition of ϕ on $\mathbb{H}_1 \oplus \mathbb{H}_2$ and the fact ϕ_{11} is *-homomorphism modulo $\mathcal{K}(\mathbb{H}_1)$, we have $\phi_{12}(a)\phi_{12}(a^*)$ is compact therefore $\phi(a)$ is compact for $a \in A$. Similarly, using $\phi(a^*a) = \phi(a)^*\phi(a)$, we have $\phi_{21}(a)$ is compact for $a \in A$. It follows that ϕ_{22} is *-homomorphism modulo $\mathcal{K}(\mathbb{H}_2)$. □

Proposition 2.11. *$K_0(D_\Phi(A)) \cong Ext^{-1}(A)$ where Φ is a admissible representation of A on a separable Hilbert space $\mathbb{H}$.*

Proof. With the Lemma 2.9 in hand, we define the map from $K_0(D_\Phi(A))$ to $Ext^{-1}(A)$ as follows
$$[p]_0 \mapsto [\tau_{\Phi,p}]$$
When $[p] = [q]$, as we have seen in the proof of Lemma 2.9, p and q are Murray- von-Neumann equivalent in $D_\Phi(A)$ so that partial isometry implementing this equivalence induces the equivalence between $\tau_{\Phi,p}$ and $\tau_{\Phi,q}$. Conversely, unitary equivalence between $\tau_{\Phi,p}$ and $\tau_{\Phi,q}$ induces Murray- von-Neumann equivalence between p and q evidently.
From $\Phi \oplus \Phi \sim_u \Phi$, we get a unitary $u \in \mathcal{B}(\mathbb{H} \oplus \mathbb{H}, \mathbb{H})$ which induces a natural isomorphism $\mathrm{Ad}(u) : \mathbb{M}_2(D_\Phi(A)) \to D_\Phi(A)$. Note $\pi \circ Ad(u) = Ad(u) \circ (\pi \otimes id_2)$. Since $[p] + [q] = [p \oplus q]$ and $p \oplus q$ is in $D_{\Phi\oplus\Phi}(A)$, $[p] + [q]$ is mapped to $[\pi \circ Ad(u) \circ (\Phi \oplus \Phi)_{p\oplus q}] = [Ad(u) \circ ((\pi \otimes id_2) \circ (\Phi \oplus \Phi)_{p\oplus q}]$ which is indeed $[\tau_{\Phi,p}] + [\tau_{\Phi,q}]$. So far we have shown the map is a monomorphism. It is remained to show the map is onto.
Suppose ρ is semi-split extension of A with a completely positive lifting $\psi : A \to \mathcal{B}(\mathbb{H})$. By Steinspring's dilation theorem, there is a non-degenerate *-representation $\phi : A \to \mathcal{B}(\mathbb{H}_0)$ and an isometry $V : \mathbb{H} \to \mathbb{H}_0$ such that $\psi(a) = V^*\phi(a)V$ for all $a \in A$. If we set $P_1 = VV^*$ and $P_2 = 1 - P_1$, then $\mathbb{H}_0 = P_1(\mathbb{H}_0) \oplus P_2(\mathbb{H}_0) = \mathbb{H}_1 \oplus \mathbb{H}_2$. If we decompose ϕ on $\mathbb{H}_0 = \mathbb{H}_1 \oplus \mathbb{H}_2$, we have $V\psi(a)V^* = VV^*\phi(a)VV^* = P_1\phi(a)P_1 = \phi_{11}(a)$. Since $\dot{\psi} = \rho$ is (injective) *-homomorphism, we

can conclude ϕ_{11} is (injective)*-homomorphism modulo compact. By the Lemma 2.10, $\phi_{12}(a)$, $\phi_{21}(a)$ are compacts for $a \in A$ and ϕ_{22} is *-homomorphism modulo compact. This implies that $[P_1, \phi] \in \mathcal{K}$. Thus $\dot{\phi}_{11}$ is an abstract Toeplitz extension. Viewing $V : \mathbb{H} \to \mathbb{H}_1$ as an unitary, we can also see that ρ is unitarily equivalent to $\dot{\phi}_{11}$. Hence we finish the proof. □

Similarly, we are going to define the map from $K_1(D(A))$ to $KK(A, \mathbb{C})$. We begin with the following lemma which is expected as we have Lemma 2.9.

Lemma 2.12. *Let A be as above. For any $\alpha \in K_1(D(A))$, there exists an unitary $u \in D(A)$ such that $\alpha = [u]_1$.*

Proof. Assume $T \in \mathbb{M}_n(D_\Phi(A)) \approx D_{\underbrace{\Phi \oplus \Phi \oplus \cdots \oplus \Phi}_{n}}(A)$ is a unitary which represents $\alpha \in K_1(D_\Phi(A))$. Let V be as above and $S = \begin{pmatrix} V & 1 - VV^* \\ 0 & V^* \end{pmatrix} \in \mathbb{M}_{2n}(D_\Phi(A))$. It is easy to check that $VTV^* + 1 - VV^* = \begin{pmatrix} vTv^* & 0 \\ 0 & 1 \end{pmatrix}$ and $S \begin{pmatrix} T & 0 \\ 0 & 1 \end{pmatrix} S^* = \begin{pmatrix} vTv^* + 1 - VV^* & 0 \\ 0 & 1 \end{pmatrix}$. Therefore $[T]_1 = [vTv^*]_1$. It is left to the reader to check that $vTv^* \in D(A)$ is also unitary. □

Proposition 2.13. *$KK(A, \mathbb{C}) \cong K_1(D_\Phi)$ where Φ is an admissible representation of unital separable C^*-algebra A on a separable Hilbert space $\mathbb{H}$.*

Proof. With the Lemma 2.12, we define the map from $K_1(D_\Phi)$ to $KK(A, \mathbb{C})$ as follows.

$$[u]_1 \mapsto \left[\left(\hat{\mathbb{H}}, \begin{pmatrix} \Phi & 0 \\ 0 & \Phi \end{pmatrix}, \begin{pmatrix} 0 & u^* \\ u & 0 \end{pmatrix} \right) \right]$$

where $\hat{\mathbb{H}}$ is a graded Hilbert $\mathbb{C}$-module $\mathbb{H} \oplus \mathbb{H}$ with the standard even grading. (See Chapter14.2 in [Bl]). Indeed, this construction gives rise to well-defined group homomorphism. If $[u] = [v]$, then $u \oplus 1$ is homotopic to $v \oplus 1$ up to unitary.
Therefore $\left(\hat{\mathbb{H}} \oplus \hat{\mathbb{H}}, \begin{pmatrix} \Phi \oplus \Phi & 0 \\ 0 & \Phi \oplus \Phi \end{pmatrix}, \begin{pmatrix} 0 & u^* \oplus 1 \\ u \oplus 1 & 0 \end{pmatrix} \right)$
$= \left(\hat{\mathbb{H}}, \begin{pmatrix} \Phi & 0 \\ 0 & \Phi \end{pmatrix}, \begin{pmatrix} 0 & u^* \\ u & 0 \end{pmatrix} \right) \oplus \left(\hat{\mathbb{H}}, \begin{pmatrix} \Phi & 0 \\ 0 & \Phi \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right)$ is operator homotopic to $\left(\hat{\mathbb{H}}, \begin{pmatrix} \Phi & 0 \\ 0 & \Phi \end{pmatrix}, \begin{pmatrix} 0 & v^* \\ v & 0 \end{pmatrix} \right) \oplus \left(\hat{\mathbb{H}}, \begin{pmatrix} \Phi & 0 \\ 0 & \Phi \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right)$

$= \left(\hat{\mathbb{H}} \oplus \hat{\mathbb{H}}, \begin{pmatrix} \Phi \oplus \Phi & 0 \\ 0 & \Phi \oplus \Phi \end{pmatrix}, \begin{pmatrix} 0 & v^* \oplus 1 \\ v \oplus 1 & 0 \end{pmatrix}\right)$. Similarly, it can be shown the map is group homomorphism. If we show it is surjection, we are done. We will use Higson's idea in p354 [Hig]. Let $\alpha \in KK(A, \mathbb{C})$ be represented by $\left(H_0 \oplus H_1, \begin{pmatrix} \phi_0 & 0 \\ 0 & \phi_1 \end{pmatrix}, \begin{pmatrix} 0 & u^* \\ u & 0 \end{pmatrix}\right)$ where u is a unitary in $\mathcal{B}(H_0, H_1)$. Let $\Psi = \cdots \phi_0 \oplus \phi_0 \oplus \phi_1 \oplus \phi_1 \cdots$ and $\mathbf{H} = \cdots H_0 \oplus H_0 \oplus H_1 \oplus H_1 \cdots$. We consider a degenerate cycle

$$\left(\mathbf{H} \oplus \mathbf{H}, \begin{pmatrix} \Psi & 0 \\ 0 & \Psi \end{pmatrix}, \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}\right)$$

Then

$$\left(H_0 \oplus H_1, \begin{pmatrix} \phi_0 & 0 \\ 0 & \phi_1 \end{pmatrix}, \begin{pmatrix} 0 & u^* \\ u & 0 \end{pmatrix}\right) \oplus \left(\mathbf{H} \oplus \mathbf{H}, \begin{pmatrix} \Psi & 0 \\ 0 & \Psi \end{pmatrix}, \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}\right)$$

is unitarily equivalent to $\left(\mathbf{H} \oplus \mathbf{H}, \begin{pmatrix} \Psi & 0 \\ 0 & \Psi \end{pmatrix}, \begin{pmatrix} 0 & F^* \\ F & 0 \end{pmatrix}\right)$ where $F = \begin{pmatrix} I & 0 & 0 \\ 0 & u & 0 \\ 0 & 0 & I \end{pmatrix} \circ$ shifting to the right i.e. F sends $(\cdots, \eta_1, \eta_0, \xi_0, \xi_1, \cdots)$ to $(\cdots, \eta_2, \eta_1, u\eta_0, \xi_0, \cdots)$.

Again by adding a degenerate cycle $\left(\hat{\mathbb{H}}, \begin{pmatrix} \Phi & 0 \\ 0 & \Phi \end{pmatrix}, \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}\right)$, we get

$$\alpha = \left[\left(\mathbf{H} \oplus \mathbb{H}, \begin{pmatrix} \Psi \oplus \Phi & 0 \\ 0 & \Psi \oplus \Phi \end{pmatrix}, \begin{pmatrix} 0 & F^* \oplus I \\ F \oplus I & 0 \end{pmatrix}\right)\right]$$

Since Φ is *admissible*, we obtain a unitary $U \in \mathcal{B}(\mathbf{H} \oplus \mathbb{H}, \mathbb{H})$ such that $Ad(U) \circ \Psi \oplus \Phi \sim \Phi$.

$$= \left[\left(\hat{\mathbb{H}}, (Ad(U) \circ \Psi \oplus \Phi) \oplus (Ad(U) \circ \Psi \oplus \Phi), \begin{pmatrix} 0 & Ad(U)(F^* \oplus I) \\ Ad(U)(F \oplus I) & 0 \end{pmatrix}\right)\right]$$

By the lemma 4.1.10. in [JenThom]

$$= \left[\left(\hat{\mathbb{H}}, \Phi \oplus \Phi, \begin{pmatrix} 0 & Ad(U)(F^* \oplus I) \\ Ad(U)(F \oplus I) & 0 \end{pmatrix}\right)\right]$$

It is not hard to check that $Ad(U)(F \oplus I) \in D_\Phi(A)$ and α is the image of it. So we finish the proof. $\square$

Remark 2.14. A unital C^*-algebra A is said to have K_1-surjectivity if the natural map from $U(A)/U_0(A)$ to $K_1(A)$ is surjective and is said to have (strong)K_0-surjectivity if the group $K_0(A)$ is generated by $\{[p] \mid p \text{ is a projection in } A\}$. Therefore Lemma 2.9 and Lemma 2.12 show $D_\Phi(A)$ has (strong)K_0-surjectivity and K_1-surjectivity.

Now we are ready to define the index pairing between $K_i(A)$ and $K_{i+1}(D_\Phi(A))$ for all $i = 0, 1$. For the following two definitions, we mean Index as the classical *Fredholm index*.

Given a projection $p \in \mathbb{M}_k(A)$ and $u \in K_1(D_\Phi(A))$, when Φ_k is k-th amplification of Φ, the operator

$$\Phi_k(p)u^{(k)}\Phi_k(p) : \Phi_k(p)(\mathbb{H}^k) \to \Phi_k(p)(\mathbb{H}^k)$$

is essentially unitary, and therefore *Fredholm*.

Definition 2.15. The (even)index pairing $K_0(A) \times K_1(D_\Phi(A)) \to \mathbb{Z}$ is defined as follows.

$$([p],[u]) \longrightarrow \text{Index}\left(\Phi_k(p)u^{(k)}\Phi_k(p)\right)$$

where $p \in \mathbb{M}_k(A)$ and Φ_k is k-th amplification of Φ.

Similarly, given $v \in \mathbb{M}_k(A)$ and $p \in K_0(D_\Phi(A))$, the operator

$$p^{(k)}\Phi_k(v)p^{(k)} - (1 - p^{(k)}) : \mathbb{H}^k \to \mathbb{H}^k$$

is essentially unitary, and therefore *Fredholm*.

Definition 2.16. The (odd)index pairing $K_1(A) \times K_0(D_\Phi(A)) \to \mathbb{Z}$ is defined as follows.

$$([v],[p]) \longrightarrow \text{Index}\left(p^{(k)}\Phi_k(v)p^{(k)} - (1 - p^{(k)})\right)$$

if $v \in \mathbb{M}_k(A)$ and Φ_k is k-th amplification of Φ.

3. Kasparov Product and Duality

In this section, we prove main results: Each index pairing is a special case of Kasparov product. Before doing this, we need some elementary computations of Kasparov groups.

Proposition 3.1. *$KK(S, B) = K_1(B)$ where S is $\{f \in C(\mathbb{T}) \mid f(1) = 0\}$ and B is a C^*-algebra.*

Proof. Most of proof can be found in [Lee]. We just note that any unitary in $K_1(B)$ can be liftable to $\boldsymbol{\phi} \in KK(S, B)$ here $\phi : S \mapsto B$ which is determined by sending $z - 1$ to u-1. □

Theorem 3.2. *(Even case)the mapping $K_0(A) \times K_1(D_\Phi) \to \mathbb{Z}$ is the Kasparov product $KK(S, SA) \times KK(SA, S) \to \mathbb{Z}$.*

Proof. Without loss of generality, we may assume p is the element of A.(If necessary, consider $\mathbb{C}^k \otimes A$) Using the K-theory Bott map, p is mapped to $f_p(z) \in K_1(SA)$. Then as we have noted in Proposition 3.1, $f_p(z)$ is lifted to $\boldsymbol{\Psi}$ as the element of $KK(S, SA)$ where Ψ is the

*-homomorphism from S to SA which is determined by sending $z-1$ to $(z-1)p$.
On the other hand, $[u] \in K_1(D_\Phi(A))$ is mapped to $[\mathcal{E}]$ in $KK(A,\mathbb{C})$ where $\mathcal{E} = \Big(\hat{\mathbb{H}}, \left(\begin{smallmatrix} \Phi & 0 \\ 0 & \Phi \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & u^* \\ u & 0 \end{smallmatrix}\right)\Big)$ is a Kasparov A-$\mathbb{C}$ module by Proposition 2.13.
Using natural isomorphism $\tau_S : KK(A,\mathbb{C}) \mapsto KK(SA,S)$, we can think of a Kasparov product $\mathbf{\Psi}$ by $[\tau_S(\mathcal{E})]$.
Using elementary functorial properties, we can check $\mathbf{\Psi}\cdot[\tau_S(\mathcal{E})]$ is equal to $\Big[(\hat{\mathbb{H}} \otimes S, (\Phi\oplus\Phi)\otimes id \circ \Psi, G\otimes id)\Big]$ denoting $\left(\begin{smallmatrix} 0 & u^* \\ u & 0 \end{smallmatrix}\right)$ by G.
Since $\tau_S : KK(\mathbb{C},\mathbb{C}) \mapsto KK(S,S)$ is isomorphism, there is a map $\rho : \mathbb{C} \mapsto \mathcal{B}(\mathbb{H})$ s.t. $\tau_S((\hat{\mathbb{H}}, \rho\oplus\rho, G)) = (\hat{\mathbb{H}}\otimes S, (\Phi\oplus\Phi)\otimes id\circ\Psi, G\otimes id)$.
Note that

$$\rho\otimes id(z-1) = (z-1)\rho(I)$$

On the other hand, from the equality above

$$\begin{aligned} &= \Phi\otimes id(\Psi(z-1)) \\ &= \Phi\otimes id((z-1)p) \\ &= (z-1)\Phi(p) \end{aligned}$$

Consequently, $\rho(I) = \Phi(p)$.
Then $\Big(\hat{\mathbb{H}}, \rho\oplus\rho, G\Big)$ is mapped to $\mathrm{Ind}(\Phi(p)u\Phi(p))$ by the map Index : $KK(\mathbb{C},\mathbb{C}) \to \mathbb{Z}$. (If you want more details, see Example 17.3.4 in [Bl] or Exercise 2.1.2 in [JenThom].) $\square$

Similarly, we have

Theorem 3.3. *(Odd case) The mapping $K_1(B)\times K_0(D_\Phi) \to \mathbb{Z}$ is the Kasparov product $KK(S,B)\times KK^1(B,\mathbb{C}) \to \mathbb{Z}$.*

Proof. Again, $[u]$ is mapped to $\boldsymbol{\psi} \in KK(S,B)$ here $\psi : S \mapsto B$ which is determined by sending $z-1$ to $u-1$. On the other hand, $[p]$ is mapped to $[\tau_{\Phi,p}]$ by Proposition 2.11. Using the isomorphism $Ext^{-1}(B) \to KK^1(B,\mathbb{C})$ (See Propsition 3.3.11 in [JenThom]), image of $\tau_{\Phi,p}$ is a cycle $\mathcal{F} = (\mathbb{H}\oplus\mathbb{H}, \Phi\oplus\Phi, T\oplus -T)$ where $T = 2p-1$. Then Kasparov product $\boldsymbol{\psi}$ by $[\mathcal{F}]$ is $\boldsymbol{\psi}\cdot[\mathcal{F}] = [(\mathbb{H}\oplus\mathbb{H}, \Phi\circ\psi\oplus\Phi\circ\psi, T\oplus -T)]$. Note that $\Phi\circ\psi(z-1) = \Phi((u-1)) = \Phi(u)-1$. Using the identification $KK^1(S,\mathbb{C}) \to K_1(\mathcal{Q}(\mathbb{H}))$, it is mapped to $p\Phi(u)p-(1-p)$. (See Proposition 17.5.7 in [Bl].) Finally, the *Fredholm index* of $p\Phi(u)p - (1-p)$ is what we want. $\square$

Corollary 3.4. *Let $\boldsymbol{x} \in KK(\mathbb{C}_1, S) \cong Ext(\mathbb{C}, S)$ be represented by the extension $0 \to S \to C \to \mathbb{C} \to 0$ and $\boldsymbol{y} \in KK(S,\mathbb{C}_1) \cong Ext(S,\mathbb{C})$ be*

represented by the extension $0 \to \mathcal{K} \to C^*(v-1) \to S \to 0$ *where* v *is a coisometry of Fredholm index 1 (e.g. the adjoint of the unilateral shift). Then* $\boldsymbol{x} \cdot \boldsymbol{y} = 1_{\mathbb{C}_1}$

Proof. Note that $\boldsymbol{x}$ corresponds to the unitary $t \mapsto e^{2\pi i t}$ in $K_1(S)$ by the Brown's Universal Coefficient Theorem [Br]. Also, the Busby invariant of $0 \to \mathcal{K} \to C^*(v-1) \to S \to 0$ is the homomorphism $\tau : S \to \mathcal{Q}$ sends $e^{2\pi i t} - 1$ to $\pi(v) - 1$. Since $KK(\mathbb{C}_1, \mathbb{C}_1) \cong \mathbb{Z}$, using Theorem3.3, we can conclude $\boldsymbol{x} \cdot \boldsymbol{y} = 1_{\mathbb{C}_1}$. □

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